TUT 5: SOLVE A HIGHER ORDER LINEAR EQUATION

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Q1. Solve the following higher order linear equations.

(a) $y^{(4)} + 2y'' + y = 3t + \cos 2t$;

(b) $y''' - y'' - y' + y = 2e^{-t} + 3$.

Solution. (a) Firstly consider the homogeneous equation, the corresponding characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2 = 0.$$

The roots are

$$\lambda_1 = \lambda_2 = i, \;\; \lambda_3 = \lambda_4 = -i.$$

Hence the fundamental set of solutions is

$$S := \{\sin t, \, \cos t, \, t \sin t, \, t \cos t\}$$

Secondly, we need to find a particular solution $Y(t) = Y_1(t) + Y_2(t)$ with

$$\begin{cases} Y_1^{(4)} + 2Y_1'' + Y_1 = 3t; \\ Y_2^{(4)} + 2Y_2'' + Y_2 = \cos 2t. \qquad \cos 2t \notin S \end{cases} \tag{1}$$

Take

$$Y_1 = A_1 t + A_2; \qquad Y_2(t) = B_1 \cos 2t + B_2 \sin 2t.$$

It is easy to see $Y_1 = 3t$ and

$$Y_2'' = -4B_1 \cos 2t - 4B_2 \sin 2t, \quad Y_2^{(4)} = 16B_1 \cos 2t + 16B_2 \sin 2t.$$

Substitute into (1), we have

$$Y_2^{(4)} + 2Y_2'' + Y_2 = (16 - 8 + 1)B_1 \cos 2t + (16 - 8 + 1) \sin 2t = \cos 2t.$$

So $B_1 = 1/9, B_2 = 0$ and then

$$Y_2 = \frac{1}{9} \cos 2t.$$

In conclusion, the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t + 3t + \frac{1}{9} \cos 2t.$$

(b) The characteristic equation is

$$\lambda^3 - \lambda^2 - \lambda + 1 = (\lambda - 1)^2 (\lambda + 1) = 0.$$

Hence the fundamental set of solutions is

$$S := \{e^t, \; te^t, \; e^{-t}\}.$$

Now we find a particular solution $Y(t) = Y_1(t) + Y_2(t)$ with

$$\begin{cases} Y_1''' - Y_1'' - Y_1' + Y_1 = 2e^{-t}; & e^{-t} \in S \\ Y_2''' - Y_2'' - Y_2' + Y_2 = 3. \end{cases} \tag{2}$$

Clearly $Y_2 = 3$, now we take $Y_1 = Ate^{-t}$, then

$$Y_1' = A(1-t)e^{-t}, \quad Y_1'' = A(t-2)e^{-t}, \quad Y_1''' = A(1-t)e^{-t}.$$

Hence

$$Y_1''' - Y_1'' - Y_1' + Y_1 = A\big[1 - t - (t-2) - (1-t) + t\big]e^{-t} = 2e^{-t},$$

which implies $A = 1$ and

$$Y_1 = te^{-t}.$$

The general solution is

$$y(t) = c_1e^t + c_2te^t + c_3e^{-t} + te^t + 3$$

for some constants c_1, c_2, c_3.

□

Q2. Determine a suitable form for a particular solution if the method of undermined coefficients is to be used.

(a) $y^{(4)} - y''' - y'' + y' = t^3 + 2te^t$;

(b) $y^{(4)} + 8y'' + 16y = \sin 2t + t\cos 2t + te^t + e^t \sin t \cos 2t$.

Solution. (a) The characteristic equation is

$$\lambda^4 - \lambda^3 - \lambda^2 + \lambda = \lambda(\lambda - 1)^2(\lambda + 1) = 0.$$

Then the fundamental set of solutions is

$$S := \{1, e^t, te^t, e^{-t}\}.$$

A particular solution is given by $Y(t) = Y_1(t) + Y_2(t)$ with

$$\begin{cases} Y_1''' - 2Y_1'' + Y_1' = t^3, \\ Y_2''' - 2Y_2'' + Y_2' = 2te^t. \end{cases} \tag{3}$$

Take

$$Y_1(t) = A_1t^4 + A_2t^3 + A_3t^2 + A_4t + 0$$

and

$$Y_2(t) = (B_1t + B_2)t^2e^t.$$

(b) The characteristic equation is

$$\lambda^4 + 8\lambda^2 + 16 = (\lambda^2 + 4)^2 = 0.$$

Then the fundamental set of solutions is

$$S := \{\cos 2t, \sin 2t, t\cos 2t, t\sin 2t\}.$$

A particular solution is given by $Y(t) = Y_1(t) + Y_2(t)$ with

$$\begin{cases} Y_1^{(4)} + 8Y_1'' + 16Y_1 = \sin 2t, \\ Y_2^{(4)} + 8Y_2'' + 16Y_2 = t\cos 2t, \\ Y_3^{(4)} + 8Y_3'' + 16Y_3 = te^t, \\ Y_4^{(4)} + 8Y_4'' + 16Y_4 = e^t \sin 2t \cos 2t. \end{cases} \tag{4}$$

We should take

$$\begin{cases} Y_1(t) = t^2(A_1 \sin 2t + A_2 \cos 2t), \\ Y_2(t) = t^2(B_1 \cos 2t + B_2 \sin 2t)(B_3 t + B_4), \\ Y_3(t) = (C_1 t + C_2)e^t, \\ Y_4(t) = e^t(D_1 \sin 4t + D_2 \cos 4t). \end{cases} \tag{5}$$

□

Q3. Let $g(t)$ be a general function of t, find a formula involving integrals for a particular solution $Y(t)$ to the differential equation

$$y''' - 3y'' + 3y' - y = g(t). \tag{6}$$

If $g(t) = t^{-2}e^t$, determine the particular solution $Y(t)$.

Solution. The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0.$$

Hence the fundamental solutions are

$$y_1 = e^t, \quad y_2 = te^t, \quad y_3 = t^2 e^t.$$

We are finding a particular solution in the form

$$Y(t) = u_1(t)y_1 + u_2(t)y_2 + u_3(t)y_3,$$

where $u_i(t)(i = 1, 2, 3)$ solve

$$\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ g(t) \end{pmatrix}.$$

Direct calculations give

$$\begin{aligned} &y_1 = y_1' = y_1'' = e^t; \\ &y_2' = (1+t)e^t, \quad y_2'' = (2+t)e^t; \\ &y_3' = (t^2 + 2t)e^t, \quad y_3'' = (t^2 + 4t + 2)e^t. \end{aligned}$$

We firstly compute the Wronskian

$$
\begin{aligned}
W(t) &= \begin{vmatrix} e^t & te^t & t^2e^t \\ e^t & (1+t)e^t & (t^2+2t)e^t \\ e^t & (2+t)e^t & (t^2+4t+2)e^t \end{vmatrix} = e^{3t} \begin{vmatrix} 1 & t & t^2 \\ 1 & 1+t & t^2+2t \\ 1 & 2+t & t^2+4t+2 \end{vmatrix} \\
&= e^{3t} \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 2 & 4t+2 \end{vmatrix} = 2e^{3t} \qquad \text{Use Abel's theorem to check your calculation}
\end{aligned}
$$

Crame's law implies that for $i = 1, 2, 3$,

$$
u_i' = \frac{W_i}{W} = \frac{1}{2} e^{-2t} W_i, \tag{7}
$$

where W_i are given as follows.

$$
\begin{aligned}
W_1(t) &= \begin{vmatrix} 0 & te^t & t^2e^t \\ 0 & (1+t)e^t & (t^2+2t)e^t \\ g(t) & (2+t)e^t & (t^2+4t+2)e^t \end{vmatrix} = g(t)e^{2t} \begin{vmatrix} 0 & t & t^2 \\ 0 & 1+t & t^2+2t \\ 1 & 2+t & t^2+4t+2 \end{vmatrix} \\
&= g(t)e^{2t}t^2;
\end{aligned}
$$

$$
W_2(t) = \begin{vmatrix} e^t & 0 & t^2e^t \\ e^t & 0 & (t^2+2t)e^t \\ e^t & g(t) & (t^2+4t+2)e^t \end{vmatrix} = g(t)e^{2t} \begin{vmatrix} 1 & 0 & t^2 \\ 1 & 0 & t^2+2t \\ 1 & 1 & t^2+4t+2 \end{vmatrix} = -2tg(t)e^{2t};
$$

and

$$
W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (1+t)e^t & 0 \\ e^t & (2+t)e^t & 1 \end{vmatrix} = g(t)e^{2t} \begin{vmatrix} 1 & t & 0 \\ 1 & 1+t & 0 \\ 1 & 2+t & 1 \end{vmatrix} = g(t)e^{2t}.
$$

Recall the identy (7), one has

$$
\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \frac{1}{2} g(t) e^{-t} \begin{pmatrix} t^2 \\ -2t \\ 1 \end{pmatrix}.
$$

Furthermore, integrating implies

$$
u_1 = \frac{1}{2} \int g(t) e^{-t} t^2 \, \mathrm{d}t, \quad u_2 = -\int g(t) e^{-t} t \, \mathrm{d}t, \quad u_3 = \frac{1}{2} \int g(t) e^{-t} \, \mathrm{d}t. \tag{8}
$$

When $g(t) = t^{-2} e^t$,

$$
u_1 = \frac{1}{2} \int 1 \, \mathrm{d}t = \frac{t}{2}, \quad u_2 = -\int t^{-1} \, \mathrm{d}t = -\ln t, \quad u_3 = \frac{1}{2} t^{-2} \, \mathrm{d}t = -t^{-1}.
$$

Hence

$$
Y(t) = \frac{1}{2} te^t - te^t \ln t - te^t = -te^t \ln t - \frac{1}{2} te^t.
$$

□

Q4. Verify $y = e^t$ is a particular solution to

$$(2-t)y''' + (2t-3)y'' - ty' + y = 0, \qquad t > 2.$$

Then use reduction method to solve the above equation.

Hint: Take $y(t) = u(t)e^t$ and deduce a equation of $u(t)$ then solve it. Indeed,

$$u''' + \frac{3-t}{2-t}u'' = 0,$$

Observe that $u = t$ is a solution, and this is an first order linear equation of $v := u''$.